HW #3 AME 567

Problem 5: | Most computations are done with matlab and or maple. Intermediate steps are often omitted.

a) the segments of a composite parametric cubic curve is given by

$$\vec{r}_i(t) = [1\ t\ t^2\ t^3] \begin{bmatrix}1&0&0&0\\0&0&1&0\\-3&3&-2&-1\\2&-2&1&1\end{bmatrix}\begin{bmatrix}\vec{r}_i^{\,T}\\ \vec{r}_{i+1}^{\,T}\\ \dot{\vec{r}}_i^{\,T}\\ \dot{\vec{r}}_{i+1}^{\,T}\end{bmatrix} \quad (*) \quad t=[0,1]$$

If the derivatives of the starting and ending vectors are specified, the derivatives at the interval points can be solved from:

$$\begin{bmatrix}1&0&0&0&0&\cdots&0\\1&4&1&0&0&&\vdots\\0&1&4&1&0&&\vdots\\ \vdots&&&&&&\vdots\\ \vdots&&&&1&4&1\\0&\cdots&&&0&0&1\end{bmatrix}\begin{bmatrix}\dot{\vec{r}}_0\\ \dot{\vec{r}}_1\\ \vdots\\ \dot{\vec{r}}_{n-1}\\ \dot{\vec{r}}_n\end{bmatrix} = \begin{bmatrix}\dot{\vec{r}}_0\\ 3(\vec{r}_2-\vec{r}_0)\\ \vdots\\ 3(\vec{r}_n-\vec{r}_{n-2})\\ \dot{\vec{r}}_n\end{bmatrix} \quad (**)$$

However $\dot{\vec{r}}_0$ and $\dot{\vec{r}}_n$ are now unspecified and we have to apply other constraints to solve for $\dot{\vec{r}}_0$ and $\dot{\vec{r}}_n$. Now the second derivatives at the beginning and ending point are zero:

$$\frac{d^2}{dt^2}\left(r(t)\right)=0 \Rightarrow \frac{d^2}{dt^2}\left(\vec{a}_0+\vec{a}_1 t+\vec{a}_2 t^2+\vec{a}_3 t^3\right) = 2\vec{a}_2+6\vec{a}_3 t = 0$$

At the begin point $t=0 \Rightarrow \vec{a}_2 = 0 \Rightarrow 3[\vec{r}(1)-\vec{r}(0)] - 2\dot{\vec{r}}(0) - \dot{\vec{r}}(1) = 0$

$$\Rightarrow 3[\vec{r}_1-\vec{r}_0] - 2\dot{\vec{r}}_0 - \dot{\vec{r}}_1 = 0 \Rightarrow \boxed{2\dot{\vec{r}}_0 - \dot{\vec{r}}_1 = 3(\vec{r}_1-\vec{r}_0)}$$ Constraint 1

At the end point $t=1 \Rightarrow 2\vec{a}_2 + 6\vec{a}_3 = 0 \Rightarrow \vec{a}_2 + 3\vec{a}_3 = 0 \Rightarrow$

$$\underbrace{\left\{3[\vec{r}(1)-\vec{r}(0)]-2\dot{\vec{r}}(0)-\dot{\vec{r}}(1)\right\}}_{\vec{a}_2} + 3\underbrace{\left\{2[\vec{r}(0)-\vec{r}(1)]+\dot{\vec{r}}(0)+\dot{\vec{r}}(1)\right\}}_{\vec{a}_3} = 0 \Rightarrow$$

$$3\vec{r}_n - 3\vec{r}_{n-1} - 2\dot{\vec{r}}_{n-1} - \dot{\vec{r}}_n + 6\vec{r}_{n-1} - 6\vec{r}_n + 3\dot{\vec{r}}_{n-1} + 3\dot{\vec{r}}_n = 0$$

$$-3\vec{r}_n + 3\vec{r}_{n-1} + \dot{\vec{r}}_{n-1} + 2\dot{\vec{r}}_n \Rightarrow \boxed{\dot{\vec{r}}_{n-1} + 2\dot{\vec{r}}_n = 3(\vec{r}_n - \vec{r}_{n-1})}$$ Constraint 2

(1.5)

Replacing the first and last row of (**) with these constraints we get

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 4 & 1 & 0 & 0 & & \\ 0 & 1 & 4 & 1 & 0 & & \\ 0 & 0 & 1 & 4 & 1 & & \\ \vdots & & & & & & \\ & & & & 1 & 4 & 1 \\ 0 & \cdots & & & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} \dot{\vec{r}}_0 \\ \vdots \\ \\ \\ \\ \dot{\vec{r}}_n \end{bmatrix} = \begin{bmatrix} 3(\vec{r}_1 - \vec{r}_0) \\ 3(\vec{r}_2 - \vec{r}_0) \\ \vdots \\ \\ 3(\vec{r}_n - \vec{r}_{n-2}) \\ 3(\vec{r}_n - \vec{r}_{n-1}) \end{bmatrix}$$

This equation can be solved to obtain the derivatives $\dot{\vec{r}}_0, \ldots, \dot{\vec{r}}_n$. Using these derivatives and the curve points $\vec{r}_0, \ldots \vec{r}_n$ the segments can be generated using (*)

b)

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} \dot{r}_0^T \\ \dot{r}_1^T \\ \dot{r}_2^T \\ \dot{r}_3^T \end{bmatrix} = 3 \begin{bmatrix} 3-2 & 2-1 & 4-2 \\ 4-2 & 3-1 & 8-2 \\ 2-3 & 1-2 & 12-4 \\ 2-4 & 1-3 & 12-8 \end{bmatrix} = 3 \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 6 \\ -1 & -1 & 8 \\ -2 & -2 & 4 \end{bmatrix}$$

Using maple:

$$\begin{bmatrix} \dot{r}_0^T \\ \dot{r}_1^T \\ \dot{r}_2^T \\ \dot{r}_3^T \end{bmatrix} = \begin{bmatrix} 4/5 & 4/5 & \frac{22}{15} \\ 7/5 & 7/5 & 46/15 \\ -2/5 & -2/5 & 64/15 \\ -14/5 & -14/5 & 58/15 \end{bmatrix}$$

we can now compute the coefficients $a_0, a_1, a_2, a_3$ for each segment using (*)

Last segment:

$$\begin{bmatrix} a_0^T \\ a_1^T \\ a_2^T \\ a_3^T \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} r_2^T \\ r_3^T \\ \dot{r}_2^T \\ \dot{r}_3^T \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & 8 \\ 2 & 1 & 12 \\ -2/5 & -2/5 & 64/15 \\ \frac{-14}{5} & \frac{-14}{5} & 58/15 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 8 \\ -2/5 & -2/5 & 64/15 \\ -12/5 & -12/5 & \frac{-2}{5} \\ \frac{4}{5} & \frac{4}{5} & \frac{2}{15} \end{bmatrix}$$

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problem 5b continued

And thus for the last segment

$$r(t) = \vec{a}_0 + \vec{a}_1 t + \vec{a}_2 t^2 + \vec{a}_3 t^3$$

$$r(t) = \begin{bmatrix} 4 \\ 3 \\ 8 \end{bmatrix} + \begin{bmatrix} -2/5 \\ -2/5 \\ 64/15 \end{bmatrix} t + \begin{bmatrix} -12/5 \\ -12/5 \\ -2/5 \end{bmatrix} t^2 + \begin{bmatrix} 4/5 \\ 4/5 \\ 2/15 \end{bmatrix} t^3 = \begin{bmatrix} 4 \\ 3 \\ 8 \end{bmatrix} + \begin{bmatrix} -0.4 \\ -0.4 \\ 4.3 \end{bmatrix} t + \begin{bmatrix} -2.4 \\ -2.4 \\ -0.4 \end{bmatrix} t^2 + \begin{bmatrix} 0.8 \\ 0.8 \\ 0.133 \end{bmatrix} t^3$$

+25

differentiating wrt time:

$$\dot{r}(t) = \vec{a}_1 + 2\vec{a}_2 t + 3\vec{a}_3 t^2 \quad \text{For } t = 1/2 \quad \dot{r}(0.5) = \vec{a}_1 + \vec{a}_2 + \frac{3}{4}\vec{a}_3 = \boxed{\begin{bmatrix} -2.2 \\ -2.2 \\ 3.9662 \end{bmatrix} = \dot{r}(0.5)}$$

$$\approx |\dot{r}(0.5)| = \sqrt{\dot{r}_x^2 + \dot{r}_y^2 + \dot{r}_z^2} = 5.0413$$

now the tangent vector is $\hat{T} = \frac{\dot{r}}{|\dot{r}|} = \boxed{\begin{bmatrix} -0.4364 \\ -0.4364 \\ 0.7868 \end{bmatrix} = \hat{T}(0.5)}$

we know $\frac{d\hat{T}}{ds} = k\hat{n}$ but $\frac{d\hat{T}}{ds} = \frac{d}{dt}(\hat{T})\frac{dt}{ds} = \frac{1}{|\dot{r}|}\frac{d}{dt}\left(\frac{\dot{r}}{|\dot{r}|}\right)$

$$\Rightarrow \frac{d\hat{T}}{ds} = k\hat{n} = \frac{1}{|\dot{r}|}\left[\frac{\ddot{r}}{|\dot{r}|} + \dot{r}\frac{d}{dt}\left(\frac{1}{|\dot{r}|}\right)\right]$$

intermezzo: $\frac{d}{dt}(|\dot{r}|^2) = \frac{d}{dt}(\dot{r}\cdot\dot{r}) = 2\dot{r}\cdot\ddot{r}$

acceleration component in direction $\hat{T}$

$$\frac{d}{dt}(|\dot{r}|^2) = 2|\dot{r}|\frac{d}{dt}(|\dot{r}|) \longrightarrow \boxed{\frac{d}{dt}(|\dot{r}|) = \frac{\dot{r}\cdot\ddot{r}}{|\dot{r}|} = \hat{T}\cdot\ddot{r}}$$

Thus $\frac{d}{dt}\left(\frac{1}{|\dot{r}|}\right) = \frac{-1}{|\dot{r}|^2}\frac{d}{dt}(|\dot{r}|) = \frac{-1}{|\dot{r}|^2}\hat{T}\cdot\ddot{r} = -\frac{\dot{r}\cdot\ddot{r}}{|\dot{r}|^3}$ (A)

$$k\hat{n} = \frac{\ddot{r}}{|\dot{r}|^2} - \frac{\dot{r}\cdot\ddot{r}}{|\dot{r}|^4}\dot{r} = \boxed{\frac{1}{|\dot{r}|^2}\left[\ddot{r} - (\hat{T}\cdot\ddot{r})\hat{T}\right] = k\hat{n}} \quad (1)$$

- differentiating twice wrt $t$ we get:

$$\ddot{r} = 2\vec{a}_2 + 6\vec{a}_3 t \longrightarrow \ddot{r}(t=0.5) = 2\vec{a}_2 + 3\vec{a}_3 = \boxed{\begin{bmatrix} -2.4 \\ -2.4 \\ -0.4 \end{bmatrix} = \ddot{r}(0.5)}$$

- plug values for $\dot{r}$, $\hat{T}$, $\ddot{r}$ into (1) and obtain.

$$k\hat{N} = \begin{bmatrix} -0.0639 \\ -0.0639 \\ -0.0708 \end{bmatrix} \Rightarrow \boxed{k = 0.1148} \quad \boxed{\hat{N} = \begin{bmatrix} -0.5564 \\ -0.5564 \\ -0.6172 \end{bmatrix}}$$

- First derive an expression for $\tau$

start with

$$\frac{d^3}{dt}(\vec{r})\cdot\hat{B} = \hat{B}\cdot\frac{d^2}{dt^2}\left(\hat{T}|\dot{r}|\right) = \hat{B}\cdot\frac{d}{dt}\left[\frac{d}{dt}(\hat{T})|\dot{r}| + \hat{T}\frac{d}{dt}\left(|\dot{r}|\right)\right] \quad \text{using}$$

$$= \hat{B}\cdot\frac{d}{dt}\left[|\dot{r}|^2 k\hat{N} + \hat{T}(\hat{T}\cdot\ddot{r})\right]$$

$$= \hat{B}\cdot\left[\frac{d}{dt}\left(|\dot{r}|^2 k\right)\hat{N} + |\dot{r}|^2 k\frac{d}{dt}(\hat{N}) + \frac{d}{dt}(\hat{T})(\hat{T}\cdot\ddot{r}) + \hat{T}\frac{d}{dt}(\hat{T}\cdot\ddot{r})\right]$$

$$= \hat{B}\cdot|\dot{r}|^2 k\frac{d}{dt}(N) + \hat{B}\cdot(\hat{T}\cdot\ddot{r})\frac{d}{dt}(\hat{T}) \qquad \text{using } \frac{d}{ds}(\hat{N}) = -k\hat{T} - \tau\hat{B}$$

$$= |\dot{r}|^3 k\,\hat{B}\cdot(-k\hat{T} - \tau\hat{B}) + (\hat{T}\cdot\ddot{r})\,\cancel{\hat{B}\cdot k\hat{N}}^{\,0}$$

$$= -|\dot{r}|^3 k\,\tau \Rightarrow \boxed{\tau = \frac{-\hat{B}\cdot\dddot{r}}{|\dot{r}|^3 k}}$$

- $\hat{B} = \hat{T}\times\hat{N} \Rightarrow$

$$\boxed{\tau = \frac{-(\hat{T}\times\hat{N})\cdot\dddot{\vec{r}}}{|\dot{r}|^3 k}} \quad (2)$$

$$\dddot{r} = 6\vec{a}_3 = \begin{bmatrix} 4.8 \\ 4.8 \\ 0.8 \end{bmatrix}$$ plugging this and the other known values in (2) obtain

$$\boxed{\tau = 0}$$